

# Radially homothetic spacetime is of Petrov-type D

Sanjay M. Wagh<sup>1</sup> and Keshlan S. Govinder<sup>2</sup>

<sup>1</sup>*Central India Research Institute, Post Box 606,  
Laxminagar, Nagpur 440 022, India  
E-mail: cirinag@nagpur.dot.net.in*

<sup>2</sup>*School of Mathematical & Statistical Sciences,  
University of Natal, Durban 4041, South Africa.  
E-mail: govinder@nu.ac.za  
(Dated: September 9, 2002)*

It is well-known [1] that *all* black hole solutions of General Relativity are of Petrov-type D. It may thus be expected that the spacetime of *physically realizable* spherical gravitational collapse is also of Petrov-type D. We show that a radially homothetic spacetime, *ie*, a spherically symmetric spacetime with hyper-surface orthogonal, radial, homothetic Killing vector, is of Petrov-type D. As has been argued in [3], it is a spacetime of *physically realizable* spherical collapse.

PACS numbers: 04.20.-q, 04.20.Cv

Keywords: Gravitational collapse - spherical symmetry - radial homothety - Petrov-type D

## I. INTRODUCTION

In General Relativity, all black hole solutions are of Petrov-type D [1]. Then, it may be expected that the spacetime of “physically realizable” spherical collapse of matter is of Petrov-type D.

In a recent work [2], we obtained a spherically symmetric spacetime by considering a metric separable in co-moving coordinates and by imposing a relation of pressure  $p$  and density  $\rho$  of the barotropic form  $p = \alpha\rho$  where  $\alpha$  is a constant. Such a relation determined only the temporal metric functions of that spacetime.

Therefore, although the temporal metric functions for this spacetime were determined in [2] using the above barotropic equation of state, same metric functions are determinable from *any* relation of pressure and density.

This spacetime admits a hyper-surface orthogonal, radial, Homothetic Killing Vector (HKV). (See later.) Hence, it will be called a *radially homothetic spacetime*. Since this spacetime describes appropriate “physical” stages of evolution of spherical matter, we have argued in [3] that it is the spacetime of *physically realizable* spherical collapse of matter. As we show in this letter, it is a Petrov-type D spacetime.

## II. SPACETIME METRIC

In general, a HKV captures [4] the notion of the scale-invariance of the spacetime. A homothetic spacetime then admits an appropriate homothetic Killing vector  $\mathbf{X}$  satisfying

$$\mathcal{L}_{\mathbf{X}}g_{ab} = 2\Phi g_{ab} \quad (1)$$

where  $\Phi$  is an arbitrary constant.

If, in terms of the chosen coordinates,  $\mathbf{X}$  has component only in the direction of one coordinate, the Einstein field equations separate for that coordinate, generating also an arbitrary function of that coordinate. This is the broadest (Lie) sense of the scale-invariance leading not only to the reduction of the field equations as partial differential equations to ordinary differential equations but leading also to their separation.

A spherically symmetric spacetime has only one spatial scale associated with it - the radial distance scale. Therefore, for a *radially homothetic spacetime*, the metric admits one arbitrary function of the radial coordinate. We then obtain for a radially homothetic spacetime arbitrary radial characteristics for matter. That is, due to the radial scale-invariance of the spherical spacetime, matter has *arbitrary* radial properties in a radially homothetic spacetime.

A general spherically symmetric spacetime admits, in co-moving coordinates, a metric of the form

$$ds^2 = -e^{2\nu(r,t)}dt^2 + e^{2\lambda(r,t)}dr^2 + Y^2(r,t)d\Omega^2 \quad (2)$$

where  $d\Omega^2 = [d\theta^2 + \sin^2\theta d\phi^2]$ .

In what follows, we therefore demand that a spherically symmetric spacetime of (2) admits a spacelike HKV of the form

$$X^a = (0, X^1(r, t), 0, 0) \quad (3)$$

Then, the expression (1) reduces to the system of four equations:

$$X^1 \nu_{,r} = \Phi \quad (4)$$

$$X^1_{,t} = 0 \quad (5)$$

$$\left(\frac{Y_{,r}}{Y} - \nu_{,r}\right) X^1 = 0 \quad (6)$$

$$(\lambda_{,r} - \nu_{,r}) X^1 + X^1_{,r} = 0 \quad (7)$$

where a comma denotes a derivative and  $\Phi$  is a constant. (See [5], for the comprehensive and detailed discussion of the general conformal geometry of the metric (2).)

Solving the above system of equations for  $X^1$ ,  $\nu$ ,  $\lambda$  and  $Y$ , we obtain

$$X^1 = F(r) \quad (8)$$

$$Y = \tilde{g}(t) \exp\left(\int \frac{\Phi}{F(r)} dr\right) \quad (9)$$

$$\lambda = \int \frac{\Phi}{F(r)} dr - \log F(r) + \tilde{h}(t) \quad (10)$$

$$\nu = \int \frac{\Phi}{F(r)} dr + \tilde{k}(t) \quad (11)$$

where  $F(r)$  is an arbitrary function of the co-moving radial coordinate  $r$ . We emphasize here that the HKV (3) is expected, on the basis of Lie's theory of differential equations, to generate an arbitrary function of the radial coordinate  $r$  for the spacetime. This is seen to be the case.

Then, the spacetime metric becomes *separable* in co-moving coordinates and is given by

$$ds^2 = k^2(t) \exp\left(2 \int \frac{\Phi}{F(r)} dr\right) \left[ -dt^2 + \frac{h^2(t)}{F^2(r)} dr^2 + g^2(t) d\Omega^2 \right] \quad (12)$$

Now, writing  $F(r) = y(r)\Phi/y'$ , we obtain

$$ds^2 = -y^2 dt^2 + \gamma^2 (y')^2 B^2 dr^2 + y^2 Y^2 d\Omega^2 \quad (13)$$

with a prime indicating a derivative with respect to  $r$ ,  $B \equiv B(t)$ ,  $Y \equiv Y(t)$  and  $\gamma = 1/\Phi$  is a constant. (Temporal function in  $g_{tt}$  is absorbed by suitable redefinition of the time coordinate.)

Therefore, the imposition of HKV (3) *uniquely* determines, in co-moving coordinates, the spherically symmetric metric to (13) [2].

Now, the radial scale-invariance of a spherical spacetime identifies the spacetime metric uniquely. To see this, let us use a gaussian radial coordinate and a area radial coordinate.

In a gaussian coordinate system, the most general spherical metric is [6]

$$ds^2 = -e^{2\nu(\bar{r},\tau)} d\tau^2 + d\bar{r}^2 + \bar{r}^2 f^2(\bar{r},\tau) d\Omega^2 \quad (14)$$

The radial scale-invariance then demands the existence of a HKV of the form  $(0, X^1(\bar{r},\tau), 0, 0)$ . Then, (1) implies

$$\nu_{,\bar{r}} X^1 = \Phi \quad (15)$$

$$X^1_{,\tau} = 0 \quad (16)$$

$$X^1_{,\bar{r}} = \Phi \quad (17)$$

$$\left(\frac{1}{\bar{r}} + \frac{f_{,\bar{r}}}{f}\right) X^1 = \Phi \quad (18)$$

The spacetime metric of a radially homothetic, spherical spacetime in a gaussian coordinate system is then uniquely obtained as

$$ds^2 = -(\Phi\bar{r} + K_1)^2 k^2(\tau) d\tau^2 + d\bar{r}^2 + (\Phi\bar{r} + K_1)^2 h^2(\tau) d\Omega^2 \quad (19)$$

where  $K_1$  is a constant of integration from (17).

Further, for the curvature or the area coordinates, the general spherical metric is [6]

$$ds^2 = -e^{2\nu(\bar{r},\bar{t})} d\bar{t}^2 + e^{2\lambda(\bar{r},\bar{t})} d\bar{r}^2 + \bar{r}^2 d\Omega^2 \quad (20)$$

The radial scale-invariance then requires us to impose the HKV  $(0, X^1(\bar{r},\bar{t}), 0, 0)$ . Then, (1) becomes

$$\nu_{,\bar{r}} X^1 = \Phi \quad (21)$$

$$X^1_{,\bar{t}} = 0 \quad (22)$$

$$X^1_{,\bar{r}} + \lambda_{,\bar{r}} = \Phi \quad (23)$$

$$\frac{X^1}{\bar{r}} = \Phi \quad (24)$$

We then, uniquely, obtain the metric

$$ds^2 = -\bar{r}^2 h^2(\bar{t}) d\bar{t}^2 + g^2(\bar{t}) d\bar{r}^2 + \bar{r}^2 d\Omega^2 \quad (25)$$

As can be easily verified, the three forms above, namely, (13), (19) and (25), are diffeomorphically equivalent to each other.

We select the metric form (13) in co-moving coordinates for our present study since it explicitly displays the radial scale-invariance of the spacetime under consideration here.

Therefore, a radially homothetic, spherical spacetime admits a spacelike HKV of the form

$$X^a = (0, \frac{y}{\gamma y'}, 0, 0) \quad (26)$$

and the spacetime metric is then given by (13).

Now, the spacetime of (13) is required, by definition, to be locally flat at all of its points including the center.

The condition for the center to possess a locally flat neighborhood is

$$y'|_{r \sim 0} \approx 1/\gamma \quad (27)$$

This condition must be imposed on any  $y(r)$ . With this condition, (27), the HKV of metric (13) is, at the center,  $y|_{r=0} \partial/\partial r$ .

Now,  $y(r)$  is the "area radius" in (13). When  $y|_{r=0} \neq 0$ , the orbits of the rotation group  $SO(3)$

do not shrink to zero radius at the center for (13). Consequently, the center is not regular for (13) when  $y|_{r=0} \neq 0$  although the curvature invariants remain finite at the center.

Also, when  $y|_{r=0} = 0$ , the center is regular for the spacetime of (13). But, the curvature invariants blow up at the center, then.

It is well-known [7] that the center and the initial data for matter, both, are not *simultaneously* regular for a spherical spacetime with hyper-surface orthogonal HKV. Therefore, the spacetime of (13) does not possess a *regular* center and *regular* matter data, simultaneously.

However, the lack of regularity of the center of (13) for non-singular matter data is understandable [3] since the orbits of the rotation group do not shrink to zero radius for every observer. It is a relative conception and the co-moving observer, it being a “cosmological” observer, of (13) is not expected to observe the orbits of the rotation group shrink to zero radius.

Further, the spacetime of a spherical body must possess non-vanishing central value for mass in it. In Newtonian gravity, this is the theorem: “The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shell’s matter were concentrated into a point at its center.”

Now, the mass function for the spacetime of (13) can be defined as

$$m(r, t) = \frac{yY}{2} \left( 1 - \frac{Y^2}{\gamma^2 B^2} + \dot{Y}^2 \right) \quad (28)$$

where  $m(r, t)$  denotes the total mass in the spacetime, ie, mass of matter together with the “effective” contribution due to the flux of radiation or heat in the spacetime.

Then, all points, including the center, of the “initial” spacelike hyper-surface, at  $t = 0$ , evolve along the respective timelike trajectories and, with all points of the hyper-surface, non-vanishing mass exists also at the center. This is consistent with the Newtonian theorem mentioned earlier.

### III. SINGULARITIES AND DEGENERACIES OF THE METRIC (13)

The Ricci scalar for (13) is:

$$\mathcal{R} = \frac{4\dot{Y}\dot{B}}{y^2 Y B} + \frac{2\ddot{B}}{y^2 B} - \frac{6}{y^2 \gamma^2 B^2} + \frac{2}{y^2 Y^2} + \frac{2\dot{Y}^2}{y^2 Y^2} + \frac{4\ddot{Y}}{y^2 Y} \quad (29)$$

The Einstein tensor for (13) is:

$$G_{tt} = \frac{1}{Y^2} - \frac{1}{\gamma^2 B^2} + \frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{B}\dot{Y}}{BY} \quad (30)$$

$$G_{rr} = \frac{\gamma^2 B^2 y'^2}{y^2} \left[ -2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y} + \frac{3}{\gamma^2 B^2} - \frac{1}{Y^2} \right] \quad (31)$$

$$G_{\theta\theta} = -Y\ddot{Y} - Y^2\frac{\ddot{B}}{B} - Y\frac{\dot{Y}\dot{B}}{B} + \frac{Y^2}{\gamma^2 B^2} \quad (32)$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta} \quad (33)$$

$$G_{tr} = 2\frac{\dot{B}y'}{By} \quad (34)$$

where an overhead dot denotes a time-derivative.

Now, for the co-moving observer with four-velocity  $\mathbf{U} = \frac{1}{y} \frac{\partial}{\partial t}$ , the *radial* velocity of the fluid is  $V^r = \dot{Y}$ . The co-moving observer is accelerating for (13) since  $\dot{U}_a = U_{a;b}U^b$  is, in general, non-vanishing for  $y' \neq 0$ . The expansion is  $\Theta = \frac{1}{y} \left( \frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y} \right)$ . Further,  $\dot{B}$  is related to the flux of radiation in the co-moving frame.

Clearly, we may use the function  $y(r)$  in (13) as a new radial coordinate - the area coordinate - as long as  $y' \neq 0$ . However, the situation of  $y' = 0$  represents a coordinate singularity that is similar to, for example, the one on the surface of a unit sphere where the analogue of  $y$  is  $\sin\theta$  [6]. The curvature invariants do not blow up at locations for which  $y' = 0$ .

Genuine curvature singularities exist for (13) when either  $y(r) = 0$  for some  $r$  or when the temporal functions vanish for some  $t = t_s$ .

There are, therefore, two types of curvature singularities of the spacetime of (13), namely, the first type for  $B(t_s) = 0$  and, the second type for  $y(r) = 0$  for some  $r$ .

Note that the “physical” radial distance corresponding to the “coordinate” radial distance  $\delta r$  is

$$\ell = \gamma(y')B\delta r \quad (35)$$

Then, collapsing matter forms the spacetime singularity in (13) when  $B(t) = 0$  is reached for it at some  $t = t_s$ . Therefore, the singularity of first type is a singular hyper-surface for (13).

The singularity of the second type is a singular sphere of coordinate radius  $r$ . The singular sphere reduces to a singular point for  $r = 0$  that is the center of symmetry. For  $y(r) = 0$  for some range of  $r$ , there is a singular thick shell. Singularities of the second type constitute a part of the initial data, singular data.

The metric (13) has evident degeneracies when  $y(r) = \text{constant}$  for some range of the co-moving

radial coordinate  $r$  or globally. Further, the metric (13) is also degenerate for  $y(r) = \infty$  either on a degenerate sphere of coordinate radius  $r$ , for some “thick shell” or globally. The degeneracy  $y(r) = \infty$  is equivalent to vacuum. For  $y(r) = \text{constant}$ , the degeneracy corresponds to uniform density. For  $y(r) = 0$ , the degeneracy is also an infinite density singularity.

*In what follows, we shall assume, unless stated explicitly, that there are no singular initial-data and that there are no degenerate situations for the metric (13).*

#### IV. SPACETIME OF THE METRIC (13) IS OF PETROV-TYPE D

In what follows, we use  $\sigma$  to denote the null quantities. For the metric (13), the null basis or the Newman-Penrose (NP) vectors are:

$$\sigma_1 = \ell = \frac{1}{\sqrt{2}} \left[ \frac{1}{y} \frac{\partial}{\partial t} + \frac{1}{\gamma y' B} \frac{\partial}{\partial r} \right] \quad (36)$$

$$\sigma_2 = n = \frac{1}{\sqrt{2}} \left[ \frac{1}{y} \frac{\partial}{\partial t} - \frac{1}{\gamma y' B} \frac{\partial}{\partial r} \right] \quad (37)$$

$$\sigma_3 = m = \frac{1}{\sqrt{2}yY} \left[ \frac{\partial}{\partial \theta} - i \csc \theta \frac{\partial}{\partial \phi} \right] \quad (38)$$

$$\sigma_4 = \bar{m} = \frac{1}{\sqrt{2}yY} \left[ \frac{\partial}{\partial \theta} + i \csc \theta \frac{\partial}{\partial \phi} \right] \quad (39)$$

One notes that in a null basis the only non-vanishing scalar products are  $\ell \bullet n = -1$  and  $m \bullet \bar{m} = 1$ .

Then, the corresponding null 1-forms are:

$$\sigma^1 = \frac{1}{\sqrt{2}} [ydt + \gamma y' Bdr] \quad (40)$$

$$\sigma^2 = \frac{1}{\sqrt{2}} [ydt - \gamma y' Bdr] \quad (41)$$

$$\sigma^3 = \frac{yY}{\sqrt{2}} [d\theta + i \sin \theta d\phi] \quad (42)$$

$$\sigma^4 = \frac{yY}{\sqrt{2}} [d\theta - i \sin \theta d\phi] \quad (43)$$

The non-vanishing NP Spin coefficients are, then, obtained as:

$$\alpha = -\beta = -\frac{\cot \theta}{2\sqrt{2}yY} \quad (44)$$

$$\epsilon = \frac{1}{2\sqrt{2}y} \left[ \frac{\dot{B}}{B} - \frac{\dot{Y}}{Y} \right] \quad (45)$$

$$\mu = \frac{1}{\sqrt{2}y} \left[ \frac{\dot{Y}}{Y} - \frac{1}{\gamma B} \right] \quad (46)$$

$$\gamma = \frac{1}{2\sqrt{2}y} \left[ \frac{2}{\gamma B} - \frac{\dot{B}}{B} - \frac{\dot{Y}}{Y} \right] \quad (47)$$

$$\rho = -\frac{1}{\sqrt{2}y} \left[ \frac{\dot{Y}}{Y} + \frac{1}{\gamma B} \right] \quad (48)$$

Now, the non-vanishing components of the Weyl tensor for (13) are:

$$C_{trtr} = \frac{B^2 \gamma^2 (y')^2}{3} F(t) \quad (49)$$

$$C_{t\theta t\theta} = -\frac{y^2 Y^2}{6} F(t) \quad (50)$$

$$C_{t\phi t\phi} = \sin^2 \theta C_{t\theta t\theta} \quad (51)$$

$$C_{r\theta r\theta} = \frac{B^2 \gamma^2 Y^2 (y')^2}{6} F(t) \quad (52)$$

$$C_{r\phi r\phi} = \sin^2 \theta C_{r\theta r\theta} \quad (53)$$

$$C_{\theta\phi\theta\phi} = -\frac{y^2 Y^4 \sin^2 \theta}{3} F(t) \quad (54)$$

where

$$F(t) = \frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} - \frac{1}{Y^2} - \frac{\ddot{B}}{B} + \frac{\dot{B}\dot{Y}}{BY} \quad (55)$$

As can be easily verified, the NP complex scalars

$$\Psi_0 \equiv -C_{pqrs} \ell^p m^q \ell^r m^s = 0 \quad (56)$$

$$\Psi_1 \equiv -C_{pqrs} \ell^p n^q \ell^r m^s = 0 \quad (57)$$

$$\Psi_3 \equiv -C_{pqrs} \ell^p n^q \bar{m}^r n^s = 0 \quad (58)$$

$$\Psi_4 \equiv -C_{pqrs} n^p \bar{m}^q n^r \bar{m}^s = 0 \quad (59)$$

and, hence, that both the NP-vectors  $\ell$  and  $n$  are aligned along repeated principal null directions of the Weyl tensor. The spacetime of (13) is therefore a Petrov-type D spacetime.

It is well-known [1] that the shear-free geodesic condition on both  $\ell$  and  $n$  in Petrov-type D spacetimes ensures that the NP spin coefficients  $\kappa$ ,  $\sigma$ ,  $\lambda$ ,  $\nu$  vanish as is evident from the NP spin coefficients listed earlier.

#### V. CONCLUDING REMARKS

The black-hole spacetimes of General Relativity are all of Petrov-type D [1]. It is, therefore, generally believed that a Petrov-type D spacetime is required to describe the *physically realizable* spherical gravitational collapse of matter. That this is true for spherical symmetry is what is the premise of the present paper.

It is well-known that Penrose [8] is led to the Weyl hypothesis on the basis of thermodynamical considerations, in particular, those related to the thermodynamic arrow of time. On the basis

of these considerations, we may consider the Weyl tensor to be “some” sort of measure of the *entropy* in the spacetime at any given epoch.

Then, for non-singular and non-degenerate data in (13), the Weyl tensor of (13) blows up at the singular hyper-surface of (13) but is “vanishing” at the “initial” hyper-surface since  $\dot{Y} = \dot{B} = 0$  for the “initial” hyper-surface [3].

This behavior of the Weyl tensor of (13) is in conformity with Penrose’s Weyl curvature hypothesis [8]. Thus, the spacetime of (13) has the “right” kind of thermodynamic arrow of time in it.

As a separate remark, we note that, following the works of Ellis and Sciama [9], there is an interpretation [10] of Mach’s principle, namely that there should be no source-free contributions to the metric or that there should be no source-free Weyl tensor for a Machian spacetime.

We, therefore, also note here that the vacuum is a degenerate case for (13). Then, the metric (13) has no source-free contributions and the spacetime of (13) has no source-free Weyl tensor since the data is required to be non-singular and non-degenerate for it. The spacetime of (13) is, then, Machian in this sense.

### Acknowledgments

We are grateful to Ravi Saraykar and Pradeep Muktibodh for verifying some of the calculations. Some of the reported calculations have been performed using the software SHEEP and SMW is indebted to Malcolm MacCallum for providing this useful package to him.

- 
- [1] Chandrasekhar S (1983) *The mathematical theory of black holes* (Oxford: Clarendon Press)
  - Hawking S and Ellis, G F R (1973) *The large scale structure of space-time* (Cambridge: Cambridge University Press)
  - [2] Wagh S M, Govender M, Govinder K S, Maharaj S D, Muktibodh P S and Moodley M (2001) *Class. Quantum Grav.* **18** 2147 - 2162
  - [3] Wagh S M, Saraykar R V and Govinder K S (2002) Title: *What is the spacetime of physically realizable spherical collapse?* submitted to *Physical Review Letters* **Database: gr-qc/0209001**.
  - [4] Carr B J and Coley A A (1999) *Class. Quantum Grav.* **16** R31 - R71
  - [5] Moopanar S (1997) in an unpublished doctoral thesis entitled *Conformal Motions in General Relativity* (University of Natal, Durban, South Africa) has worked out a general form of the solution of the conformal Killing equations for spherically symmetric spacetimes subject to twelve integrability conditions.
  - [6] Synge J L (1964) in *Relativity, Groups and Topology, Les Houches Lectures* (New York: Gordon and Breach)
  - Abrams L S (1979) *Phys. Rev. D* **20** 2474. Also available as **Database: gr-qc/0201044**
  - [7] Macintosh, C. B. (1975) *Gen. Rel. Grav.* **7** 199
  - [8] Penrose R (1979) in *General Relativity - An Einstein Centenary Survey* (Eds. S Hawking and W Israel, Cambridge: Cambridge University Press)
  - [9] Ellis G F R and Sciama D (1972) In *General Relativity: papers in honor of J L Synge* (Ed. O’Raifeartaigh L, Oxford: Clarendon Press)
  - [10] Tod K P (1994) *Gen. Rel. Grav.* **26** 103